Nucleon Structure, Duality and Elliptic Theta Functions

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Abstract

Nucleon structure functions are shown to have a qualitative (or 'formal') relation to the classical elliptic theta functions. In particular, $\theta_1'/2$ shows a clear resemblance to a non-singlet structure function like xF_3. In the appropriate range, the Q^2-dependence of the moments of $\theta_1'/2$ is in near-quantitative agreement with QCD, and at low-Q^2 the moments converge to a common value, as observed empirically for xF_3. At very high Q^2 ($Q \to M_{PL}$), $\theta_1'/2 \to \eta^3(x)$ (where $\eta(x)$ is the Dedekind eta function) while xF_3 in the same limit appears closer to $\eta^5(x)$. A comparison of the theta function identity $\theta_2^4(0) = \theta_3^4(0) - \theta_4^4(0)$ with the relation $xF_3 = Q(x) - \bar{Q}(x)$ suggests that singlet structure functions have more in common with θ_3 and θ_4. The possibility of interesting large-x/small-x 'duality' relations for structure functions emerges naturally from the analysis.

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Structure functions have long been recognised as fundamental by experimentalists, phenomenologists and theorists alike, and impressive progress has been made over the years in achieving a precise, quantitative understanding of their behaviour, over an increasingly large a kinematic range, mainly within the context of perturbative QCD (see for example Ref. [1]). A quantitative description/representation of structure functions to match that achieved to date in perturbative QCD, however, is not the immediate (nor indeed the ultimate) aim of the present paper, and neither in fact is QCD our point of departure here. Rather the aim of this paper is to draw attention to a rather startling (if qualitative) *empirically*-observed similarity/relation existing between nucleon structure functions and the classical elliptic (or Jacobi [2]) theta functions, almost certainly involving, it would appear, the very highest energy scales, eg. the unification scale or the Planck-energy.

Theta functions of various kinds, and in particular the modular symmetry inherent to theta functions, are themselves recognised as fundamental, mainly within the context of more-abstract theory, most notably in string-theory [3]. Triggering the present paper, and drawing strongly on considerations of modular symmetry (with specific reference to the Jacobi theta functions in the scale-invariant case) are recent results [4] on strong/weak-coupling duality, monopole condensation etc, which are generally believed to bear on the confinement problem, and which in the long-term might reasonably be expected to lead to practical calculations of relevant non-perturbative quantities in theories akin to QCD. Thus while (as we shall see) there do seem to be points of contact from our observations to perturbative QCD and to a number of associated phenomenological ideas/models, we do have to draw attention to the possibility of an important link to what one might call 'fundamental theory' here. Naturally, we continue to emphasise the empirical aspects of our observations, which we regard as striking and interesting in their own right, regardless of explanation.

With duality ideas in mind therefore, we begin by observing that the modular transformation formula (Weyl inversion) for a simple theta function $\theta[\tau]$ viz:

$$\theta[\tau] = \sqrt{\frac{i}{\tau}}\theta[-1/\tau] \tag{1}$$

is arguably just the kind of relation one might hope to establish between the strong-coupling ($\tau \to 0$) and weak-coupling ($\tau \to \infty$) domains in a theory like QCD. The parameter τ is in general complex (with Im $\tau > 0$) but for definiteness we focus here on the case that τ is purely imaginary:

$$\tau = it \qquad (t > 0) \tag{2}$$

so that $t = -i\tau$ is real and positive in this case (via the notion of the 'running' coupling, we might think of t as the logarithm of some 'energy', see below).

The four classical theta functions $\theta_i(z|\tau)$, $i = 1-4$ [2] all have well-defined modular transformation properties, each satisfying a relation similar to Eq. 1 in the case that $z = 0$. In Ref. [4] the functions θ_2, θ_3 and θ_4 (evaluated at $z = 0$) arise in connection with the analysis of spin structures in massless S-dual theories with dimensionless couplings. In this paper we make particular use of the function $\theta_1'(z|\tau)$:

$$\theta_1'(z|\tau) = 2 \sum_{n=0}^{\infty} (-1)^n e^{i2\pi\tau(2n+1)^2/8} (2n+1) \cos(2n+1)z \qquad (3)$$

obtained from θ_1 by differentiation with respect to z (the θ_i are by no means all independent, eg. $\theta_2(z) = \theta_1(z + \pi/2)$, $\theta_3(z) = \theta_4(z + \pi/2)$ and $\theta_1'(0) = \theta_2(0)\,\theta_3(0)\,\theta_4(0)$ [2], where $\theta_i(0)$ denotes a theta function evaluated at $z = 0$).

With regard to the experimental data on structure functions, theoretical fits based on perturbative QCD (which makes rather clear-cut predictions for the Q^2 dependence of structure functions, at sufficiently high Q^2) have typically been used to interpolate and extrapolate existing experimental data to cover the full kinematic range of current/future experimental interest (ie. up to and including LHC energies). By way of example, in Figure 1a we show (solid curves) the x-dependence of the isospin-averaged, non-singlet neutrino-nucleon structure function xF_3, based on the MRS [5] fit, for factor-of-ten increments in Q^2, for $Q^2 = 1 - 10^7$ GeV^2 (here Q^2 is the square of the four-momentum transfer and x is the usual Bjorken scaling variable). The data used in these fits cover the range $Q^2 = 1 - 10^4$ GeV^2, the fit being deliberately restricted to $Q^2 > 1$ GeV^2, to reduce the risk of trespass into the non-perturbative regime. While the x-dependence at fixed Q^2 is largely determined by the data in these fits (perturbative QCD has much less to say about the expected x-dependence), a number of phenomenological ideas/models do serve as some guide in selecting appropriate functional forms for fitting. Regge-theory [6], for example, suggests an $x^{\frac{1}{2}}$ dependence for xF_3 at small x and dimensional counting rules [7] suggest $(1 - x)^3$ for the behaviour as $x \to 1$, these predictions being expected to hold at some not-too-large Q^2 ($Q^2 \simeq 1$ GeV^2). The sequence of curves shown in Figure 1a prompts the question of what is the high energy limit of the structure function or (perhaps more relevantly) what is the extrapolated form at the unification scale or the Planck energy?

The undenied continued sucess of thermodynamic/statistical-models [8] [9] of structure functions would certainly suggest that statistical-mechanical (or combinatorial) considerations could underlie the form of the structure functions at high energies (it has also been noted directly from experiment that the x-dependence of structure functions becomes more exponential in character with increasing energy [10]). Closely associated with theta functions, the Dedekind eta function $\eta[\tau]$ obeys the same modular transformation formula as the simple theta function (Eq. 1 above) and has a

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clear combinatorial significance (see below). Furthermore the x-dependence of the eta function, given by:

$$\eta(x) = x^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - x^n) \qquad (4)$$

($\eta(x) \equiv \eta[\tau]$, $x = \exp(2\pi i \tau)$) is remarkably reminiscent of the x-dependence expected of a non-singlet structure function, cf. the Regge prediction etc. (above). With $t = -i\tau$ taken to be real and positive as before (cf. Eq. 2) we have:

$$t = \frac{1}{2\pi} \ln \frac{1}{x} \qquad (5)$$

with x also real and positive and satisfying $0 < x < 1$ as required. With this (perhaps unexpected) association of t with the scaled parton energy x (rather than $Q = \sqrt{Q^2}$), the modular transformation (cf. Eq. 1) would become a large-x/small-x relation, ie. a symmetry versus $\ln t = \ln \frac{1}{2\pi} \ln 1/x$ centered about a 'critical' x-value $x_0 = \exp(-2\pi)$.

In Eq. 4 the pre-factor $x^{1/24}$ is known to be essential to the modular covariance. The infinite product has a clear combinatorial significance as the reciprocal of the generating function for the classical partition function [11]. Physically it is just the reciprocal of the ordinary partition function for a system comprising a (freely) variable number of indistinguishable massless bosons moving freely in one dimension (the index 1/24 in Eq. 4 is then interpreted as the associated Casimir (or vacuum) energy [12]). The generalisation to more than one 'colour' of boson is readily obtained by raising the generating function to the power of the number of boson colours [13]. Functions like $\eta^3(x)$, $\eta^8(x)$ ... appear naturally as operator traces or 'characters' in relation to the infinite dimensional lie algebras $\hat{su}_2$, $\hat{su}_3$ ... which are equivalent to (2-dimensional) current algebras [13]. The (one-colour) generating function for fermionic partitions, related to $\eta(x)/\eta(x^2)$, is the same infinite product (Eq. 4) but with $(1+x^n)^{-1}$ replacing $(1 - x^n)$. The bosonic entropy is then $\sqrt{2}$ times the fermionic entropy and (all else being equal) bosons take twice the energy of fermions aymptotically [14].

In a heuristic spirit, we are in effect arguing that the above functions relate to the probability (per unit $\log x$) of finding the system in a particular state, and as such represent structure functions in the high energy limit, if we assume that partons behave like free particles at high energy. In Figure 2a we plot the moments of xF_3 (solid circles) for factor-of-ten increments in Q^2 (as above) calculated from the QCD fit, as a function of $\ln Q^2/\Lambda^2$ on a log-log plot, ie. we plot $\ln M_3(N)$ versus $s = \ln \ln Q^2/\Lambda^2$ (with $\Lambda \simeq 250$ MeV) so that perturbative QCD evolution is reduced to a linear dependence with a known slope ($M_3(N) \equiv \int x^{N-1} F_3 dx$ [15]). Extrapolating (the moments of) xF_3 to $Q^2 \sim 10^{38}$ GeV^2 ($Q \sim M_{PL}$), we found to reasonable accuracy: $xF_3 \rightarrow [x^{\frac{1}{12}} \eta(x)/\eta(x^2)]^5$, where the moments of this function are represented in Figure 2a by

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the open half-circles plotted at the largest value of Q^2. This function (clearly cloaely related to the fermionic partition function above) is plotted versus x in Figure 1a (dashed curve, labelled $Q^2 \sim M_{PL}^2$) and is in fact very similar in general appearance to the function $\eta^5(x)$ (not shown), either function satisfying the GLS sum-rule [16] for the $N = 1$ moment to an accuracy of a few percent.

The above observations (Figure 1a/2a) lend empirical support to the notion that powers (or combinations of powers) of $\eta(x)$ may underlie the functional form of the structure functions at the highest energies (regarding the basic shape and also, it would seem in this case, the normalisation). On the other hand, we have no particular explanation for the apparent preference for the fifth power here (but see Ref. [34] below).

The theta function θ_1' (Eq. 3) evaluated at $z = 0$ is proportional to the third power of the eta function, specifically:

$$\theta_1'(0, x)/2 = \eta^3(x). \tag{6}$$

where we adopt the usual convention $\theta_i(z, x) \equiv \theta_i(z|\tau)$ (the conventional symbol q^2 often used in place of x, is deliberately avoided here). In Figure 1b we plot the evolution of θ_1' for factor-of-ten increments in Q^2, with the a posteriori (see below) identification (assumed valid in the limit $z \to 0$):

$$z \quad \simeq \quad \text{const.} \quad - \quad \frac{1}{2\pi} \ln \ln Q^2/\Lambda^2 \qquad (z \lesssim 0.6) \tag{7}$$

where the constant is fixed such that $z = 0$ corresponds to $Q^2 \simeq 10^{38}$ GeV^2 (ie. the constant is equal to the Q^2-dependent term with Q set equal to M_{PL}, viz. const. $\simeq$ $(1/2\pi) \ln \ln M_{PL}^2/\Lambda^2$ where M_{PL} is the Planck-mass, $M_{PL} \simeq 10^{19}$ GeV). Comparing Figure 1a and Figure 1b, it is apparent that the z-dependence of the theta function shows a really rather startling similarity to the Q^2 dependence of the structure function, at least over the current experimentally relevant range $Q^2 = 1 - 10^7$ GeV^2 ($z \simeq 0.55 - 0.25$).

A more quantitative comparison is achieved by plotting (the logarithm of) the x-moments of the theta function versus z (Figure 2b) alongside the corresponding plot (Figure 2a) for the structure function moments versus $s = \ln \ln Q^2/\Lambda^2$. In each case we see a broadly linear dependence of (the logarithm of) the moments on the ordinate, with a slope which increases with increasing moment number. The theta function moments (Figure 2b) converge to a common value (2π) as $z \to \pi/2$, where $\theta_1'/2 \to 2\pi\delta(x - 1)$. While the structure function moments cannot be plotted (versus s) for $Q^2 < \Lambda^2$ and are not plotted anyway for $Q^2 < 1$ GeV^2, as explained above, it is evident nonetheless (Figure 2a) that if the linear dependence of the QCD fit were

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naively continued to lower Q^2 (broken lines) the structure function moments would also apparently converge in very similar way (this striking feature of the data has no explanation in the context of perturbative QCD, and has been cited as evidence for 'valons' [17], and is also apparently a prediction of bag-models [18] and possibly also chiral-soliton models [19]). Of course the structure function becomes proportional to $\delta(x - 1)$ at very small values of Q^2, as a consequence of the dominance of elastic scattering ($xF_3 \to 5.7 \times \delta(x - 1)$, as $Q^2 \to 0$ [15]).

Assuming the correspondence Eq. 7, in Figure 3 we compare the slopes (with respect to $\log Q^2$) of the moments of the theta function (solid curve) directly with the leading-order QCD non-singlet anomalous dimensions, evaluated for three quark flavours (broken histogram) for the first ten moments ($N = 1-10$). The theta function slopes are evaluated at some not-too-small value of z ($z = \pi/6 \simeq 0.5$), roughly where present-day measurements cluster. Somewhat remarkably the theta function slopes are seen to reproduce the three flavour anomalous dimensions to an accuracy of a few percent or so, over the range of moment numbers plotted, and it is this comparison which justifies, (a posteriori) the identification Eq. 7, where the normalising factor $1/(2\pi)$ is inserted purely empirically. At large N the non-singlet anomalous dimensions grow like $\frac{16}{27} \log N$, while the corresponding dependence for the theta function slopes appears to be $\frac{1}{\pi}\sqrt{2N}$. Large N moments are not explored experimentally, however.

Note that this remarkable quantitative agreement, whether partly (or even wholly) 'accidental', extends over the entire kinematic range explored/fitted experimentally. At higher values of Q^2 ($z \lesssim \pi/12$) the z-dependence of the theta function moments (Figure 2b) begins to 'flatten-out', with Bjorken scaling holding 'exactly' for the theta function at the very highest energies ($Q \to M_{PL}$). While this effect is naturally not present in the QCD fit (extrapolated to high energy) it is clear that such high energies ($Q^2 \gtrsim 10^7$ GeV^2) have so far also not been explored experimentally. (Certainly in any case, QCD can hardly be expected to hold unmodified for $Q \to M_{PL}$). Taking the Q^2-dependence of the theta function seriously, at the lower values of Q^2 ($z \gtrsim \pi/12$) these effects are small (see above) and better than 1% measurements of xF_3 would be required to establish them even at HERA, $Q^2 \sim 10^4$ GeV^2 [20]. By the same token, substantive deviations ($\gtrsim 20\%$) from the familiar perturbative QCD Q^2-dependence might be expected for $Q^2 \gtrsim 10^7$ GeV^2 and could conceivably show-up, for example at a next-generation ep collider.

To summarise, we have seen that the classical theta function θ_1' bears a clear resemblance to a non-singlet structure function, like xF_3. While these two functions are not identical, even in the region explored experimentally, it is hard to believe that the observed similarities are entirely accidental. We are led to conclude that a qualitative

relationship exists between structure functions and theta functions, a notion on which we will enlarge below. This is the main conclusion of our investigation and it rests on the analysis of the non-singlet structure functions, Figures 1-3 above.

Regarding singlet structure functions, ie. $F_2 = Q(x) + \bar{Q}(x)$, or the closely related gluon distribution $G(x)$ in QCD, we have only a few comments. First, we note a similarity of form between, for example, the relation $xF_3 = Q(x) - \bar{Q}(x)$ and a theta function identity like $\theta_2^4(0) = \theta_3^4(0) - \theta_4^4(0)$, essentially the Jacobi 'obscure' formula [2]:

$$16x^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 + x^n)^8 = \prod_{n=1}^{\infty} (1 + x^{n-\frac{1}{2}})^8 - \prod_{n=1}^{\infty} (1 - x^{n-\frac{1}{2}})^8 \tag{8}$$

Multiplying the theta function identity by the infinite product $\prod(1-x^n)^{24}$ and plotting the x-dependence of the various terms in Figure 4, the resulting curves are reminiscent of the familiar behaviour of $Q(x)$, $\bar{Q}(x)$ etc. at small x. This suggests that singlet structure functions may be related in some way to functions like θ_3 and θ_4 which are expressible as simple infinite products [2], without any powers of x. In this connection, we note further that, exploiting the modular covarince of Eq. 4, the infinite product $\prod(1 - x^n)$ can be shown [11] to be remarkably well approximated by:

$$\prod_{n=1}^{\infty} (1 - x^n) \quad \overset{x \to 1}{\simeq} \quad \sqrt{2\pi} \frac{x^{-\frac{1}{24}}}{\sqrt{\ln \frac{1}{x}}} \exp(-\frac{\pi^2}{6 \ln \frac{1}{x}}) \tag{9}$$

(with less than 10% deviation even for x as small as $x = 10^{-6}$ for example) where the right-hand side bears a clear resemblance to the well-known $Q^2 \to \infty$ expression for the (unintegrated) gluon distribution at small x ($x \lesssim 10^{-3}$) [21]:

$$\frac{\partial G(x)}{\partial \ln k_T^2} \quad \overset{x \to 0}{\simeq} \quad \sqrt{\frac{k_T^2}{k_1^2}} \frac{x^{-\lambda_{BFKL}}}{\sqrt{\ln \frac{1}{x}}} \exp(-\frac{\ln^2 k_T^2/k_0^2}{56\zeta(3)\bar{\alpha} \ln \frac{1}{x}}) \tag{10}$$

obtained by solving the BFKL [22] equation analytically [23] (and it will be noted that despite the 'formal' inconsistency of the limits (Eq. 9 vs. Eq. 10) the ranges of validity actually overlap over three orders of magnitude in x). In Eq. 10, $\bar{\alpha} \equiv \frac{3}{\pi}\alpha$ (where α is the QCD gauge-coupling) and $\lambda_{BFKL} = 4\ln 2 \ \bar{\alpha}$ (k_0^2, k_1^2 are non-perturbative scale parameters).

It should be pointed out that if the analytic similarity of Eq. 9 to Eq. 10 is not accidental, then there may be someting to learn here about how the BFKL singularity is regulated in QCD (Eq. 9 LHS does not obviously violate the Froissart bound). Likewise, the gauge-coupling [24] would seem to be related to the index 1/24 above (we note that for QCD, the Freudenthal de Vries 'strange' formula reduces to $8 \times C_A = 24$, where $C_A = 3$, while for three generations of quarks the corresponding net strength of

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the quark-gluon vertex in QCD is given by $18 \times C_F = 24$, where $C_F = 4/3$). Casimir himself postulated a relation between the zero-point energy coefficient and the gauge coupling [25]. In (2-dimesional) dilaton gravity, the conformal anomaly $c/24$ is known to play the role of the (dimensionless) inverse Newton constant [26].

Of course theta functions solve the diffusion equation $\partial^2\theta/\partial z^2 = (4i/\pi)\,\partial\theta/\partial\tau$ and the link between BFKL and diffusion has long been recognised, having its physical origin in the absence of transverse momentum (k_T) ordering in the BFKL ladder [27]. While the kind of global similarity to theta functions seen in the non-singlet case (Fig. 1-2) would certainly suggest that diffusion plays an important role for longitudial variables also, clearly it is not possible (from perturbative QCD alone) to derive a double-differential equation for structure functions (comparable to the diffusion equation) for arbitrary x and Q^2. In the limit $Q^2 \to \infty$, $x \to 0$ the DGLAP equation [28] for $G(x)$ in fact reduces to a wave equation: $\partial^2 G/\partial z\partial\tau \simeq 2\pi\,\partial G/\partial\tau + \ldots$[29], but it should be remembered that the DGLAP approach omits explicitly the unordered k_T contribution above (assumes strong k_T ordering). Regarding our extrapolation to $Q \to M_{Pl}$, where presumably quantum gravity effects cannot be neglected (ie. beyond the QCD context) it is perhaps worth noting that the partition function for a microscopic black-hole is known to satisfy the 1D-diffusion equation [30]. While the smooth extrapolation of DIS physics to black-hole physics implied in the present paper should obviously be viewed with due caution, we note further that a partial analogy between microscopic black-hole evaporation (via Hawking radiation [31]) and QCD fragmentation has in fact already been recognised in the literature [32] (and see also Ref. [34] below).

If anything, the above observations on singlet structure functions reinforce the case for a connection between structure functions and theta functions, which was the main conclusion of the non-singlet analysis. That is not to say that we have in hand a complete and quantitative description of the neutrino-nucleon structure functions in terms of the Jacobi theta functions. The comparisons we have made are at best only partially successful and sometimes apparently self-contradictory (eg. xF_3 cannot 'equal' θ_1' which behaves like $x^{\frac{1}{8}}$ at small x and simultaneously display an $x^{\frac{1}{2}}$ dependence as implied by Eq 8). Also some generalisation of Eq. 7, to cover the low Q^2 region, needs to be found, which may be difficult in practice. Nonetheless, considering the non-singlet and singlet cases together, the accumulated evidence for a qualitative (or 'formal') relation between structure functions and theta functions is undeniably rather strong. The most obvious possibility would be that structure functions *are* theta functions of some sort (or powers of theta functions) or some variant of theta functions with similar behaviour. Possibly, theta functions relate to some 'idealised' structure

function eg. of a pure-gauge system (or in the massless/SUSY limit) unobservable experimentally. The theoretical derivation of such a result might conceivably proceed within the representation theory of some infinite dimensional algebra [13].

To conclude, our point of view is that theta functions arise in connection with the highest energy scales, eg. the Planck-scale or the unification scale (if our observations mean anything at all, *some* high energy scale *must* be associated with the $z = 0$ for the theta functions, and the Planck/unification scale seems the obvious best guess, cf. Eq. 7). While in the QCD context we are often used to evolving models/fits from lower to higher energies, one might reasonably argue that symmetry and simplicity are in fact more naturally associated with the highest energy scales, so that ultimately, evolving functional forms *down* in energy may come to seem much more sensible. We remark once more that modular functions and modular symmetry already feature prominently in theoretical considerations of Planck-scale physics. For example, string theory threshold corrections to the reciprocal gauge-couplings [33] turn out to be in part proportional to $\ln |\eta(\tau)|^4$, (with τ the value of a given moduli field scaled to the Planck-energy). Also, in five-dimensional D-brane theory the partition function of a microscopic black-hole (small with respect to the compactification radii) can be expressed in terms of integral powers of the eta-function [34]. The possibility that such apparently abstract theoretical concerns may already be mirrored in existing phenomenology (cf. $xF_3 \rightarrow \sim \eta^5(x)$ for $Q \rightarrow M_{PL}$) is both exciting and potentially important.

For the future, emerging naturally from this analysis large-x/small-x 'duality' relations for structure functions of the form Eq. 1 (or similar) are certainly an attractive and ultimately testable possibility. While the non-singlet case would seem to offer the best chance, empirical fits to singlet structure functions (eg. to F_2^{ep} Ref. [35]) do in fact suggest 'critical' x-values of a sort, apparently Q^2-independent (with $x_0 \sim \exp(-\pi)$). With *some* correlation between the large-x and small-x behaviour of structure functions undoubtedly physically plausible [36], we have every reason to hope that definitive experimental or theoretical evidence for the existence of relations of the above form will eventually be found. In this last regard, we are pleased to call attention to the work of R. Janik [37], pointing to a connection between xF_3 and the classical elliptic theta functions, established (for transverse variables) in the Regge limit of QCD, via the modular invariance of the 'odderon'.

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Figure Captions

Figure 1 a) The x-dependence of the non-singlet neutrino-nucleon structure function xF_3, based on the standard MRS [5] QCD fit to existing data, for factor-of-ten increments in Q^2, $Q^2 = 1 - 10^7$ GeV^2 (solid curves). The dashed curve labelled $Q^2 \sim M_{\rm PL}^2$ is the function $[x^{\frac{1}{12}}\eta(x)/\eta(x^2)]^5$ (where $\eta(x)$ is the Dedekind eta function [13] Eq. 4) which provides a reasonably accurate representation of the QCD fit extrapolated to $Q^2 \sim 10^{38}$ GeV^2 (see Figure 2a). b) The theta function $\theta_1'(z,x)/2$ plotted as a function of x (solid curves) for a sequence of z-values, $z = 0.55 - 0.25$ (see Eq. 7). The limiting form of $\theta_1'/2$ at $z = 0$ is $\eta^3(x)$ (dashed curve).

Figure 2 a) (The natural logarithm of) the x-moments ($N = 1 - 7$) of the structure function xF_3, plotted as a function of $s = \ln \ln Q^2/\Lambda^2$, for $Q^2 = 1 - 10^7$ GeV^2 (filled points). Extrapolating the QCD dependence (solid lines) to $Q^2 \sim 10^{38}$ GeV^2, the extrapolated moments are seen to be reasonably accurately reproduced by the moments of the function $[x^{\frac{1}{12}}\eta(x)/\eta(x^2)]^5$ (open half-points). A naive extrapolation to low Q^2 (broken lines) shows the moments apparently converging (see text). b) The x-moments of the theta function ($\theta_1'/2$) plotted as a function of z. The theta function moments also converge to a common value as $z \to \pi/2$ (ie. at small values of Q^2) where $\theta_1'/2 \to 2\pi\delta(x - 1)$. At small z (ie. at large values of Q^2) the z-dependence 'flattens-off', so that Bjorken scaling holds exactly for θ_1' as $z \to 0$.

Figure 3 The slopes of the theta function moments with respect to $\log Q^2$, (assuming the correspondence Eq. 7) evaluated at $z = \pi/6 \simeq 0.5$, plotted as a function of moment number N (solid curve). The broken histogram shows the leading-order QCD anomalous dimensions, calculated for three quark flavours. The theta function slopes reproduce the leading-order QCD anomalous dimensions (to an accuracy of a few percent or so) over the range of moment numbers plotted.

Figure 4 Terms in the theta function identity $\theta_2^4(0) = \theta_3^4(0) - \theta_4^4(0)$ [2] (essentially the Jacobi 'obscure' formula Eq. 8) multiplied by the infinite product $\prod(1 - x^n)^{24}$, plotted as a function of x, for small-x. The resulting curves are reminiscent of the familiar behaviour of $Q(x)$, $\bar{Q}(x)$ etc. at small-x (cf. $\theta_2^4(0) = \theta_3^4(0) - \theta_4^4(0) : xF_3 = Q(x) - \bar{Q}(x)$), suggesting that singlet structure functions may be related to functions like θ_3 and θ_4, expressible as simple infinite products, without any powers of x.
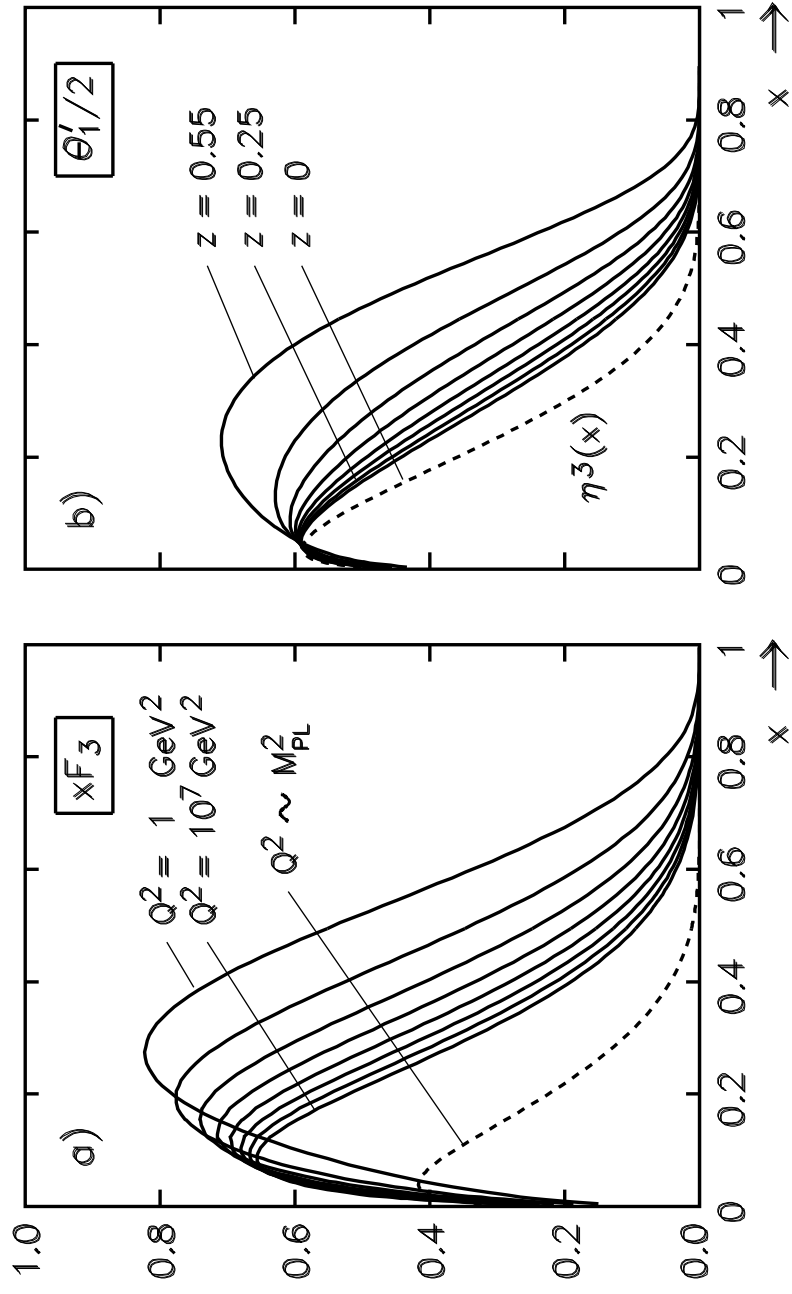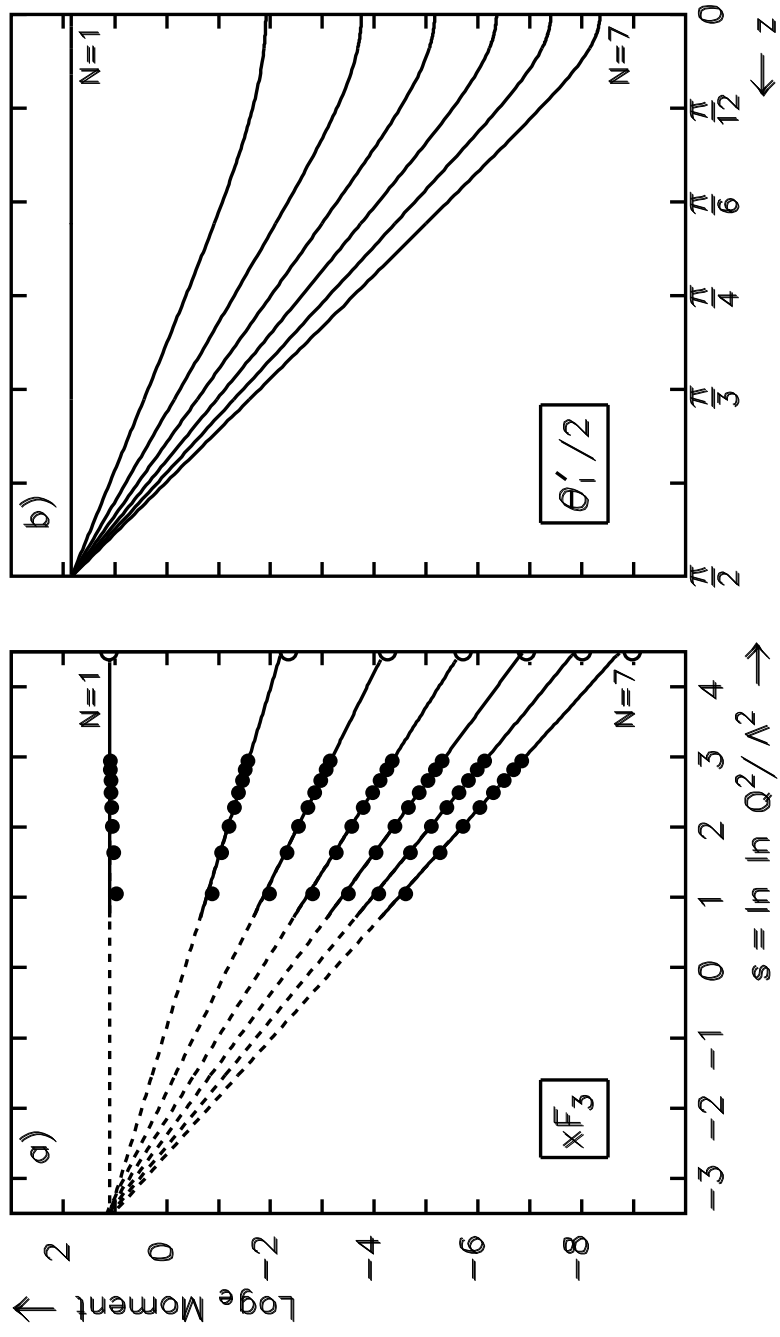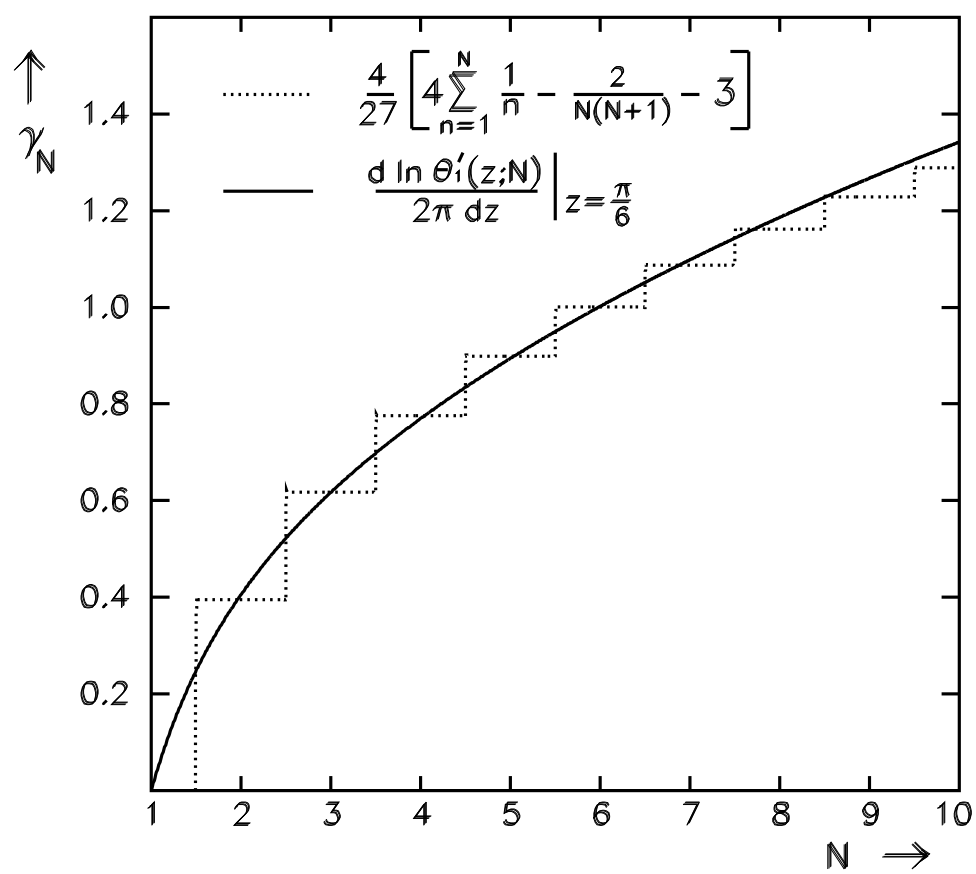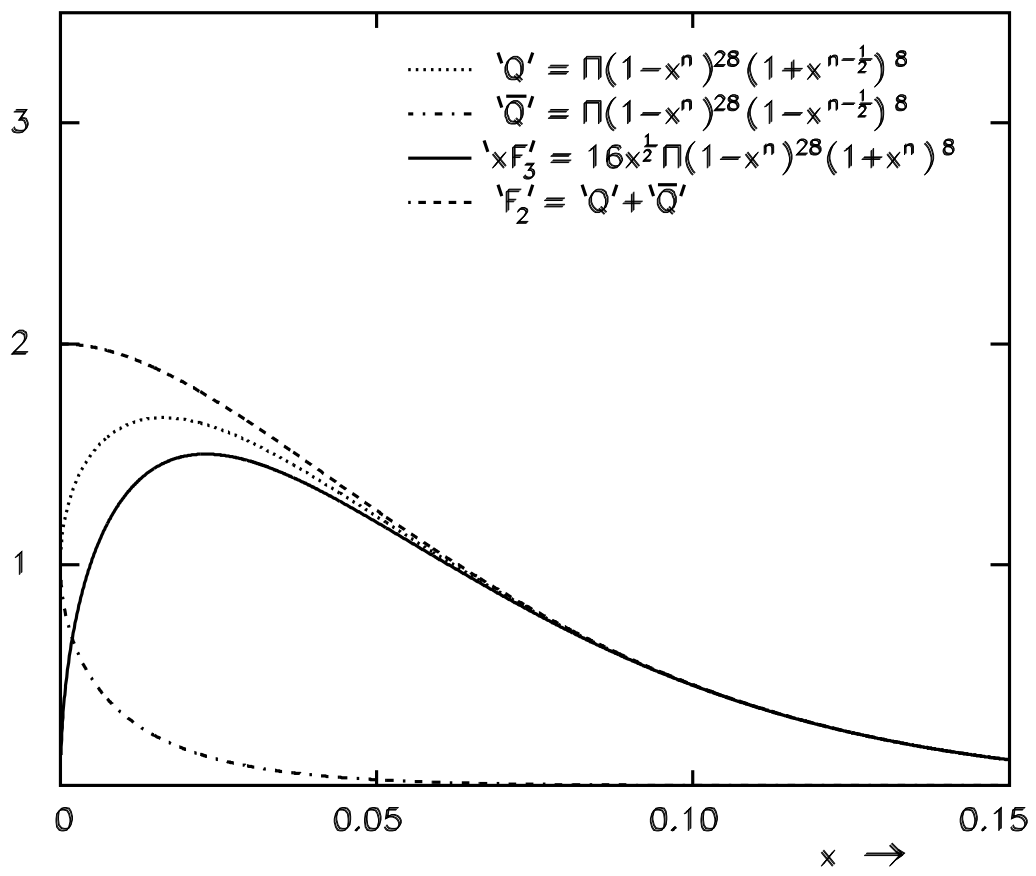
Figure 1

Figure 2

$$\cdots\cdots\quad \frac{4}{27}\left[4\sum_{n=1}^{N}\frac{1}{n}-\frac{2}{N(N+1)}-3\right]$$

$$\underline{\qquad}\quad \frac{d\,\ln\,\theta_1'(z;N)}{2\pi\,dz}\Bigg|_{z=\frac{\pi}{6}}$$

Figure 3

Figure 4